Zero-One Law for Directional Transience of One Dimensional Excited Random Walks

Gideon Y. Amir, Noam Berger, Tal Orenshtein

April 30, 2013

Abstract

The probability that a one dimensional excited random walk in stationary ergodic and elliptic cookie environment is transient to the right (left) is either zero or one. This solves a problem posed by Kosygina and Zerner [8].

1 Introduction

Excited random walk was introduced by Benjamini and Wilson in 2003 [4]. Later the model was generalized by M. P. W. Zerner [13]. It was studied extensively in recent years by numerous researchers, and an almost up to date account may be found in the recent survey of Kosygina and Zerner [8].

The generalized model due to Zerner is informally known as a cookie random walk, recently the term 'Brownie Motion' has been gaining popularity among researchers in the field.

The model is defined as follows: Let $\Omega = [0,1]^{\mathbb{Z}\times\mathbb{N}}$, and endow this space with the usual σ-algebra (namely the product of Borel σ-algebras on the intervals). Let μ be a probability measure on Ω which is invariant and ergodic w.r.t. the $\mathbb{Z}$ shift (but not necessarily the $\mathbb{N}$ shift). We call μ the cookie distribution. We call each $\omega \in \Omega$ a cookie environment. Notationally, $\omega(x,n) \in [0,1]$ is called the n-th cookie in the location x.

We say that the distribution μ is *elliptic* if it is supported on $(0,1)^{\mathbb{Z}\times\mathbb{N}}$. μ is called *uniformly elliptic* if there exists $\epsilon > 0$ such that μ is supported on $[\epsilon, 1-\epsilon]^{\mathbb{Z}\times\mathbb{N}}$.

Given a cookie environment ω and an initial position $x \in \mathbb{Z}$ we define the excited random walk driven by ω:

$$\begin{aligned} P_{\omega,x}(X_0 = x) = 1&, \\ P_{\omega,x}(X_n = X_{n-1}+1 \mid X_0, X_1, \ldots, X_{n-1}) &= \omega(X_{n-1}, \#\{k \le n-1 : X_k = X_{n-1}\}), \\ P_{\omega,x}(X_n = X_{n-1}-1 \mid X_0, X_1, \ldots, X_{n-1}) &= 1 - P_\omega(X_n = X_{n-1}+1 \mid X_0, X_1, \ldots, X_{n-1}). \end{aligned}$$

We associate μ with the annealed, or averaged, distribution defined by

$$\mathbb{P}_x = \int_\Omega P_{\omega,x} d\mu(\omega).$$

In this paper we are interested in the probability that the random walk is *transient to the right*, i.e. that $\lim_{n\to\infty} X_n = \infty$. We use A^+ to denote the event that the random walk is transient to the right, and A^- to denote the event of transience to the left (i.e. $\lim_{n\to\infty} X_n = -\infty$).

In their recent survey Kosygina and Zerner raised a version of the following problem (see Problem 3.5 of [8]):

Problem 1.1. *Find conditions on the distribution μ which imply a zero one law for a directional transience of one dimensional excited random walk, i.e. conditions which imply that $\mathbb{P}_0(A^+) \in \{0, 1\}$.*

The main result of this paper answers Problem 1.1, namely

Theorem 1.2. *Let μ be a stationary ergodic (with respect to the $\mathbb{Z}$ shift) and elliptic probability measure on the space E of cookie environments. Then $\mathbb{P}_0(A^+) \in \{0, 1\}$.*

1.1 Previous work

In some examples in the literature, special cases of Theorem 1.2 are derived as special cases of stronger characterization theorems.

Benjamini and Wilson [4] showed that whenever a cookie environment ω satisfies $\omega(x, 1) = p$ for all $x \in \mathbb{Z}$ and $\omega(x, i) = \frac{1}{2}$ for all $x \in \mathbb{Z}$ and $i \geq 2$, then the walk is $\mathbb{P}_\omega$-a.s. recurrent for all $p \in (0, 1)$. Zerner [13] showed that if the measure μ is stationary and ergodic and positive (that is, it is supported on $[\frac{1}{2}, 1]^{\mathbb{Z}\times\mathbb{N}}$), then it is right transient if and only if either $\mu(\omega(0, 1) = 1) = 1$ or $\delta > 1$, where $\delta = \mathbb{E}_\mu(\sum_{i=1}^{\infty}(2\omega(0, i) - 1))$. Kosygina and Zerner [7] showed that whenever the measure μ is i.i.d. (that is, the sequence of rows $\omega(x, \cdot)$, $x \in \mathbb{Z}$ is i.i.d. under μ), bounded (that is, there exists a deterministic number M so that $\mu(\omega(0, i) = \frac{1}{2})) = 1$ for all $i > M$), and weakly elliptic (that is, $\mu(\prod_{i=1}^{M} \omega(0, i)) > 0$ and $\mu(\prod_{i=1}^{M}(1 - \omega(0, i))) > 0$) then the walk is right transient if and only if $\delta > 1$, and left transient if and only if $\delta < -1$. Kosygina and Zerner [8] proved a Kalikow-type 0-1 law, i.e. a 0-1 law for (non directional) transience for stationary and ergodic elliptic measure over cookie environments, see Theorem 3.1.

1.2 Structure of the paper

The paper is structured as follows: In Section 2 we present some concepts and processes that take part in the proof. In section 2.1 we introduce arrow environments, in section 2.2 we introduce two associated processes Z^+ and Z^- , and in section 2.3 we study some of their connections to cookie random walks. In section 2.4 we study monotonicity and symmetry properties of Z^+ and Z^-, and present an easy lemma which is, however, the core of our argument. In Section 3 we reprove a theorem by Kosigina and Zerner. We do this for two purposes. The first purpose is to keep the paper self-contained, and the second is to enable us to easily use notations and lemmas from their proof in the proof of our main result. Finally, in Section 4 we prove Theorem 1.2.

2 Preliminaries

In this section we give some basic definitions and lemmas which are necessary for the proof of Theorem 1.2.

2.1 Arrow environments

Let ω be a cookie environment. We can materialize ω into a list of arrows, or instructions, which tell the walker where to walk to in every step of the process. More precisely, let $U = [0,1]^{\mathbb{Z}\times\mathbb{N}}$, and let $F : \Omega \times U \to \{0,1\}^{\mathbb{Z}\times\mathbb{N}}$ be defined as $F(\omega, u)(x, n) = \mathbf{1}_{u(x,n)<\omega(x,n)}$. An element $a \in \{0,1\}^{\mathbb{Z}\times\mathbb{N}}$ is called an *arrow environment*.

We now endow U with the natural σ-algebra, and with a product $\mathcal{U}[0,1]$ distribution. The following lemma is a standard Ergodic theoretic fact:

Lemma 2.1. *Consider $\{0,1\}^{\mathbb{Z}\times\mathbb{N}}$, the space of arrow environments with the standard σ-algebra, and let ν be the probability measure induced from $\Omega \times U$ on $\{0,1\}^{\mathbb{Z}\times\mathbb{N}}$ by the function F. Then ν is stationary and ergodic.*

The following fact, which is straightforward, lies behind the definition of arrow environments: (See [1] for more details.)

Fact 2.2. *Given a cookie environment ω, the law of a (non-random) walk walking according to the (ranodm) arrow environment sampled from ω is the same as the quenched law of the cookie random walk on ω.*

One may think of the arrow environments as a natural way to couple different processes on the same cookie environment. The use of arrow environments gives a direct approach to distilling the "combinatorial" part from many of the probabilistic arguments appearing in the ERW literature. See e.g. section 2.3.

2.2 The processes Z^+ and Z^-

To avoid degenerate cases we introduce the following definition.

Definition 2.3. We say that sequence of arrows $b \in \{0,1\}^{\mathbb{N}}$ is *non-degenerate* if there are infinitely many $i \geq 1$ for which $b(i) \neq b(i+1)$. An arrow environment a is called *non-degenerate* if $a(n,\cdot)$ is non-degenerate for all $n \in \mathbb{Z}$. The subspace of all non-degenerate arrow environments is denoted by $A \subset \{0,1\}^{\mathbb{Z}\times\mathbb{N}}$

Let $a \in A$ be a non-degenerate arrow environment and let $y \geq 1$. We define the processes Z^+ and Z^- (the initial value y and the sequence a is suppressed in the notation) as follows: $Z_0^+ = y$. Then, for every $n > 0$, we define Z_n^+ to be the number of 1-s until the Z_{n-1}^+-st zero in $a(n-1,\cdot)$. More precisely, if

$$\Theta_n = \inf\left\{ j : \sum_{i=0}^{j} \left[1 - a(n-1,i)\right] = Z_{n-1}^+ \right\},$$

then we take $Z_n^+ = \Theta_n - Z_{n-1}^+$.

We define Z^- completely analogously, by replacing the roles of 0 and 1, and considering a on the left half line rather than the right half line: $Z_0^- = y$, and Z_n^- is the number of 0-s until the Z_{n-1}^--st one in $a(1-n,\cdot)$.

For ease of notation, we define for every non-degenerate $b \in \{0,1\}^{\mathbb{N}}$ the functions $U_b^+, U_b^- : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ by $U_b^+(0) = 0$,

$$U_b^+(x) = \inf\left\{ j : \sum_{i=0}^{j} \left[1 - b(i)\right] = x \right\} - x, \tag{1}$$

and, defining b^c by $b^c(x) = 1 - b(x)$,

$$U_b^-(x) = U_{b^c}^+(x).$$

Using this notation, we may simply write $Z_n^+ = U_{a(n-1,\cdot)}^+(Z_{n-1}^+)$ and $Z_n^- = U_{a(1-n,\cdot)}^-(Z_{n-1}^-)$.

The definition given here for Z^+ and Z^- appeared first in [1], where the authors of that paper considered the case of any given number of walkers on the same cookie environment and used a natural generalization of the above process. A slightly different version of the chains Z^+ and Z^- was introduced and linked to one dimensional ERW by Kosygina and Zerner in 2008 [7] in the context of *bounded environments*, i.e. environments for which there is a deterministic M so that $e(x,i) = \frac{1}{2}$ for all $i > M$ and all $x \in \mathbb{Z}$. In such environments Z^+ and Z^- may be viewed as a certain type of a branching process with migration. The connection of random walks to branching process with migration is traced back at least to Kesten, Kozlov, and Spitzer [6] from 1975. An adaptation of their method to ERW was first made by Basdevant and Singh [2] in 2008. Using this connection, much can be said about the ERW, see e.g. Kosygina and Zerner [7] and [8] (transience versus recurrence, ballicticity, CLT), Basdevant and Singh [2] and [3] (ballisticity and asymptotic rate of diffusivity), Peterson [11] and [10] (law of large deviation, slow-down phenomenon, and strict monotonicity results), Rastegar and Roitershtein [12] (maximum occupation time) and Dolgopyat and Kosygina [5] and Kosygina and Mountford [9] (limit laws).

2.3 Survival of Z^+ and the hitting time T_{-1}

The aim of this section is to show that strict positivity of the process Z^+ is equivalent to the event that an excited walker on the given arrow environment never hits the point -1. This equivalence is shown to hold also for several walkers walking on the same arrow environment in [1].

Undressing the probabilistic interpretation out of the arguments of Kosigina and Zerner in Section 3 of [7], a pure combinatorial condition for right-transience is attained. Fix an arrow environment $a \in A$ and set

$$T_{-1} := \inf\{t \geq 0 : X_t = -1\}$$

to be the hitting time of -1 by the excited walk. Define

$$W_0 = 1;\ W_n = \#\{t < T_{-1} : X_t = n-1 \text{ and } X_{t+1} = n\},\ n \geq 1$$

to be the total number of right crossings of the edge $\{n-1, n\}$ by X before hitting -1.

Notice that as a is assumed to be in A, if $T_{-1} = \infty$ then $\lim_{t\to\infty} X_t = +\infty$. In particular, for each edge with non negative endpoints, the difference between the total number of its right

crossings and left crossings by the walk is exactly 1. In the case where $T_{-1} < \infty$ we have an equality. Let us sum up this fact in the following remark.

Remark 2.4. The following hold for all $n \geq 0$.

1. If $T_{-1} = \infty$ then $W_n = 1 +$ total number of left crossings of $(n, n-1)$ by X.
2. If $T_{-1} < \infty$ then $W_n =$ the total number of left crossings of $(n, n-1)$ by X before time T_{-1}.

Lemma 2.5. *For all $n \geq 0$, the following hold.*

- *if $T_{-1} < \infty$ then $Z_n^+ = W_n$.*
- *if $T_{-1} = \infty$ then $Z_n^+ \geq W_n$.*

Proof. Assume first that $T_{-1} < \infty$. We will prove by induction on $n \geq 0$ that $Z_n^+ = W_n$. For $n = 0$ we have $Z_0 = 1 = W_0$ by definition. Assume now that $Z_n^+ = W_n$. Since $T_{-1} < \infty$, the last crossing of the undirected edge $(n, n+1)$ by X before time T_{-1} is a left crossing, and therefore $a(n,i) = 0$, where i is the total number of visits of X to position n before time T_{-1}. This implies that the number of 0-s in $\{a(n,1), ..., a(n,i)\}$ equals the total number of left crossings of $(n, n-1)$ before time T_{-1}. By Lemma 2.4 the last quantity equals W_n. Since by the induction hypothesis $Z_n^+ = W_n$, We get that Z_n^+ is the number 0-s in $\{a(n,1), ..., a(n,i)\}$. Now, W_{n+1} is the number of ones in $\{a(n,1), ..., a(n,i)\}$. The latter is exactly the number of 1-s prior to Z_n^+ 0-s in $a(n, \cdot)$, which is defined to be Z_{n+1}.

Consider now the case $T_{-1} = \infty$. Again, we will prove by induction on $n \geq 0$ that $Z_n^+ \geq W_n$. For $n = 0$ we have $Z_0 = 1 = W_0$. Assume by induction that $Z_n^+ \geq W_n$. Let i be the total number of visits of X to place n. Note that as $a \in A$ the process X is transient and so $i < \infty$ and $a(n,i) = 1$. By Lemma 2.4 W_n equals the number of 0's in $\{a(n,1), ..., a(n,i)\}$ plus 1. As in the first case by the induction hypothesis Z_n^+ is greater than or equal to the number of 0's in $\{a(n,1), ..., a(n,i)\}$ plus 1. Now, W_{n+1} is the number of ones in $\{a(n,1), ..., a(n,i)\}$. The latter is exactly the number of 1-s prior to the $Z_n^+ - 1^{\text{st}}$ zero in $a(n, \cdot)$, and this number is less than or equal to Z_{n+1}^+. ☐

As a result, we get the following theorem.

Theorem 2.6. *$T_{-1} < \infty$ if and only if $Z_n^+ = 0$ for some n, and $T_1 < \infty$ if and only if $Z_n^- = 0$ for some n.*

Proof. For the first equivalence, assume first that $T_{-1} < \infty$, then $M := \max\{X_t : t < T_{-1}\} < \infty$. Therefore by Lemma 2.5 we get that $Z_{M+1}^+ = W_{M+1} = 0$. On the other hand, if $T_{-1} = \infty$, then $W_n \geq 1$ for all $n \geq 0$. Now if $Z_n^+ = 0$ then by Lemma 2.5 we have that $W_n \leq Z_n^+ = 0$, a contradiction. Considering $\bar{a}$ instead of a, where $\bar{a}(n,i) = 1 - a(-n,i)$, the argument from the beginning of the present subsection shows the second equivalence. ☐

Remark 2.7. The proof of Theorem 2.6 shows that if $M := \max\{X_t : t < T_{-1}\} \in \mathbb{N} \cup \{\infty\}$ is the maximal position of the walk before reaching -1 and $\tau = \inf\{n \geq 0 : Z_n^+ = 0\} \in \mathbb{N} \cup \{\infty\}$ is the extinction time of Z^+, then $\tau = M + 1$, where, by convention, $\infty = \infty + 1$.

2.4 Symmetry and monotonicity properties of Z^+ and Z^-

There are two immediate properties of U^+ and U^- which will be crucial for our arguments:

Observation 2.8. The following hold for any $b \in \{0,1\}^{\mathbb{N}}$.

1. $U_b^+(x)$ and $U_b^-(x)$ are nondecreasing in x.

2. $U_b^+(U_b^-(x)) < x$ for all $x \in \mathbb{Z}_+$ (and equivalently $U_b^-(U_b^+(x)) < x$).

The next symmetry lemma is simple but crucial for the proof of Theorem 1.2. For $z \in \mathbb{Z}$ we define the $\mathbb{Z}$-shift map $\theta^z : A \to A$ by $(\theta^z a)(x,i) = a(x+z,i)$, $x \in \mathbb{Z}$ and $i \in \mathbb{N}$.

Lemma 2.9 (Symmetry)**.** *Assume that for the arrow environment $a \in A$ the process Z^+ with initial value $Z_0^+ = y$ has $Z_l^+ \geq k$. Then on the shifted arrow environment $\theta^{l-1}a$, the process Z^- with initial value $Z_0^- = k$ has $Z_l^- \leq y$.*

Proof. Starting at y, and running sequently $U_{a_0}^+, \dots, U_{a_{l-1}}^+, U_{a_{l-1}}^-, \dots, U_{a_0}^-$, then by property 2 of Observation 2.8 the sequence ends below y. Denote $m = Z_l^+ \geq k$, then this means that starting at m, and running $U_{a_{l-1}}^-, \dots, U_{a_0}^-$ we end below y. By the monotonicity property 1 of Observation 2.8, we get that the same holds for k instead of m. □

3 A Kalikow type 0-1 law

The main purpose of this section is to present the proof, originally by Kosigyna and Zerner, of a Kalikow type 0-1 law, and by it set the ground for the proof of our main result, which is to be found in the next section.

Theorem 3.1 (Kosigyna-Zerner 2012)**.** *Let μ be a stationary ergodic elliptic probability measure over cookie environments. Then $\mathbb{P}_0(|X_n| \to \infty) \in \{0,1\}$.*

The proof we shall describe is the case $k = 1$ in [1], which uses an adaptation of Lemma 8 of [7].

As a convention, we say a probability measure μ over cookie environments has a given property of cookie environments if every cookie environment has it μ-a.s. We say μ satisfies a given property P of arrow environments if almost every arrow environment has the property P with respect to the annealed measure $\mathbb{P}$ associated to μ. For example μ is elliptic if $\mu((0,1)^{\mathbb{Z}\times\mathbb{N}}) = 1$, and μ is non-degenerate if the induced measure on arrow environments satisfies $\mathbb{P}(a \in A) = 1$.

Unless otherwise mentioned, from now onwards we assume that μ is a stationary ergodic elliptic probability distribution over cookie environments. As a first reduction to the proofs of both Theorem 3.1 and Theorem 1.2 we will use the following definition and results from Section 2 of Kosygina and Zerner [8], regarding the question of finiteness of the ERW range. The following lemma is elementary (see, e.g. [8] Lemma 2.2). For $x \in \mathbb{Z}$ and $\omega \in \Omega$, let $R(x,\omega)$ be the event that $\sum_{i=1}^{\infty}(\omega(x,i)) < \infty$ and $L(x,\omega)$ the event that $\sum_{i=1}^{\infty}(1-\omega(x,i)) < \infty$.

Lemma 3.2. *Assume that $\mu(R(0,\omega)) = 0, \mu(L(0,\omega)) = 0$. Then $\mathbb{P}_0$-a.s.*

$$\limsup_{n\to\infty} X_n \in \{-\infty,+\infty\} \text{ and } \liminf_{n\to\infty} X_n \in \{-\infty,+\infty\}$$

Theorem 3.3. *[Kosygina-Zerner 2012, Theorem 2.3] Let μ be a stationary and ergodic elliptic probability measure over cookie environments.*

(a) If $\mu(R(0,\omega))) > 0$ and $\mu(L(0,\omega))) > 0$ then the range is $\mathbb{P}_0$-a.s. finite.

(b) If $\mu(R(0,\omega))) = 0$ and $\mu(L(0,\omega))) > 0$ then $\mathbb{P}_0$-a.s. $X_n \to +\infty$ as $n \to \infty$.

(c) If $\mu(R(0,\omega))) > 0$ and $\mu(L(0,\omega))) = 0$ then $\mathbb{P}_0$-a.s. $X_n \to -\infty$ as $n \to \infty$.

(d) If $\mu(R(0,\omega))) = 0$ and $\mu(L(0,\omega))) = 0$ then the range is $\mathbb{P}_0$-a.s. infinite.

The proofs of Lemma 3.2 and Theorem 3.3 are straightforward, and are therefore omitted.

Corollary 3.4. *Let μ be a stationary ergodic elliptic probability measure over cookie environments. If $\mu(R(0,\omega)) > 0$ or $\mu(L(0,\omega))) > 0$ then $\mathbb{P}_0(X_n \to +\infty), \mathbb{P}_0(X_n \to -\infty) \in \{0,1\}$. In particular, in this case $\mathbb{P}_0(|X_n| \to +\infty) \in \{0,1\}$.*

Notice that a probability measure μ over cookie environments satisfies $\mu(R(x,\omega)) = \mu(L(x,\omega)) = 0$ for every $x \in \mathbb{Z}$, if and only if it is non-degenerate.

Given a walk X_n on $\mathbb{Z}$, a *right-excursion* (from 0) is a sequence of steps $X_{\tau_0}, \ldots, X_{\tau_1} \leq \infty$ of the walk such that $X_{\tau_0} = 0$, either $X_{\tau_1} = 0$ or $\tau_1 = \infty$, and $X_t > 0$ for all $\tau_0 < t < \tau_1$. Call $m \geq 0$ an *optional regeneration position* for an arrow environment a if the walk started at m never hits $m-1$, that is if $T_{m-1} = \infty$. We call $m \geq 0$ a *regeneration position* if in addition, when starting the walk from 0, the walk X_n reaches m after some finite time. Note that the arrow environment on $[m, \infty)$ remains unchanged until the walker reaches m for the first time (if it ever does). It follows that if m is an (optional) regeneration position, and the walk X_n reaches m, then afterwards it will never return to $m-1$.

Lemma 3.5. *Let μ be a stationary ergodic elliptic probability measure over cookie environments. If $\mathbb{P}_0(T_{-1} = \infty) > 0$ then there are $\mathbb{P}_0$-a.s. infinitely many optional regeneration positions.*

Proof. Let $p := \mathbb{P}_0(T_{-1} = \infty) > 0$. By stationarity of the arrow environment, $\mathbb{P}_m(T_{m-1} = \infty) = p$ for any $m \geq 0$. By the ergodic theorem, we have that

$$\frac{1}{n} \sum_{m=1}^{n} \mathbf{1}_{\{m \text{ is a optional regeneration position}\}} \to p \text{ a.s.}.$$

In particular there are a.s. infinitely many optional regeneration positions. ☐

The following lemma is a part of Lemma 8 of [7], which is proved for the the i.i.d. case.

Lemma 3.6. *Let μ be a stationary ergodic elliptic probability measure over cookie environments. Assume that $\mu(R(0,\omega)) = \mu(L(0,\omega)) = 0$. If $\mathbb{P}_0(T_{-1} = \infty) > 0$ then there are a.s. only finitely many right excursions.*

Proof. Consider $\limsup X_n$. If $\limsup X_n < \infty$, then by Lemma 3.2 $X_n \to -\infty$. In particular, the number of right excursions is finite. If $\limsup X_n = \infty$, then since by Lemma 3.5 on the event $\limsup X_n = \infty$ there a.s. exist (infinitely many) optional regeneration positions, the walk hit such a position m at some finite time and from that time on we never return to $m-1$, let alone 0, and thus there are only finitely many right excursions. ☐

Also the following lemma is a part of Lemma 8 of [7].

Lemma 3.7. *If $\mathbb{P}_{\omega,0}(T_{-1} < \infty) = 1$ and ω is elliptic, then all right excursions are $\mathbb{P}_{\omega,0}$-a.s. finite.*

Proof. The proof uses a finite modification argument which is standard (see [7] proof of Lemma 8, or [8] (3.2) and Figure 1. there). For convenience we shall supply a sketch. We will prove that the i-th right excursion is a.s. finite by induction on i. For $i = 1$ this is true by the assumption for the environment ω after its first arrival to 1. Assume now that the first i right excursions are a.s. finite and consider the past including the first step of the $(i+1)$-st excursion. The event that the last excursion is finite depends only on what the walk has done in places $x > 0$. Therefore, the probability that the $(i+1)$-st excursion is finite given that the past does not change when we modify parts of the past as long as we do not change the parts when $x > 0$. In particular it remains the same when we erase all visits to the negative integers and visits to zero are concatenated in time (simply by replacing enough of the first arrows above 0 to be right arrows). As the modified event has positive probability, conditioning on it, the probability for finiteness of the $(i+1)$-st excursion equals the probability that the first excursion is finite conditioned on making a pre-given sequence of first steps on the positive half line. This equals 1 by the assumption of the lemma since by ellipticity of ω there is a positive probability to make any pre-given finite sequence of moves. □

Corollary 3.8 (Kosigyna-Zerner 2012, AO 2013+)**.** *Let μ be a stationary ergodic elliptic probability measure over cookie environments. $\mathbb{P}_0(T_{-1} = \infty) > 0$ if and only if $\mathbb{P}_0(X_n \to +\infty) > 0$*

Proof. Note that if $T_{-1} = \infty$, then by Lemma 3.2 $\liminf X_n = +\infty$ which yields $X_n \to +\infty$. For the other implication, assume $\mathbb{P}_0(T_{-1} < \infty) = 1$, then $\mathbb{P}_{\omega,0}(T_{-1} < \infty) = 1$ for μ-a.e. $\omega \in \Omega$. By Lemma 3.7 $\mathbb{P}_{\omega,o}$-a.s. all right excursions are finite and in particular $\mathbb{P}_{\omega,o}$-a.s. $X_n \nrightarrow +\infty$ for μ-a.e. $\omega \in \Omega$. In other words, in this case $\mathbb{P}_0(X_n \to +\infty) = 0$. □

Proof of Theorem 3.1. If $\mathbb{P}_0(T_{-1} = \infty) > 0$ (resp. $\mathbb{P}_0(T_1 = \infty) > 0$) then by Lemma 3.6 there are $\mathbb{P}_0$-a.s. only finitely many right (resp. left) excursions. In particular $\mathbb{P}_0$-a.s. the walk visits 0 only finitely many times from the right (resp. left). By the assumption, for every m

$$\mathbb{P}_0\left[\prod_{i=m}^{\infty}(1 - e(0,i)) = 0\right] = 1 \quad \left(\text{resp. } \mathbb{P}_0\left[\prod_{i=m}^{\infty} e(0,i) = 0\right] = 1\right)$$

so we have $\mathbb{P}_0$-a.s. only finitely many visits to zero, which implies that $\lim_{n\to\infty} |X_n| = \infty$.

On the other hand, if $\mathbb{P}_0(T_{-1} < \infty) = 1$ and $\mathbb{P}_0(T_1 < \infty) = 1$ then by Lemma 3.7 the walk is $\mathbb{P}_0$-a.s. not right transient and not left transient. This means it is $\mathbb{P}_0$-a.s. recurrent. □

For $y, n \in \mathbb{N}$ and $B \subset \mathbb{N}$ denote by $\mathbb{P}^y(Z_n^+ \in B)$ the probability that the process Z^+ with initial value y satisfies $Z_n^+ \in B$. $\mathbb{P}^y(Z_n^- \in B)$ is defined similarly.

Let S_+ and S_- be the events that $\{Z_n^+ > 0 \text{ for all } n\}$ and $\{Z_n^- > 0 \text{ for all } n\}$, respectively.

Corollary 3.9. *Let μ be an elliptic non-degenerate probability measure over cookie environments. $\mathbb{P}_0(X_n \to +\infty) = 1$ if and only if $\mathbb{P}^1(S_+) > 0$ and $\mathbb{P}^1(S_-) = 0$.*

Proof. If $\mathbb{P}_0(X_n \to +\infty) = 1$ then by Corollary 3.8 $\mathbb{P}_0(T_{-1} = \infty) > 0$ and by Theorem 2.6 also $\mathbb{P}^1(S_+) > 0$. If by negation $\mathbb{P}^1(S_-) > 0$, then by Theorem 2.6 also $\mathbb{P}_0(T_1 = \infty) > 0$, and so by Corollary 3.8 also $\mathbb{P}_0(X_n \to -\infty) > 0$, which contradicts the assumption.

For the other direction, again by Corollary 3.8 and Theorem 2.6 we know that $\mathbb{P}_0(X_n \to +\infty) > 0$ and $\mathbb{P}_0(X_n \to -\infty) = 0$. By Theorem 3.1, we have $\mathbb{P}_0(|X_n| \to +\infty) = 1$, and so $\mathbb{P}_0(X_n \to +\infty) = 1$. □

4 Proof of the main result

This section is devoted to the proof of Theorem 1.2. We will first state a key proposition.

Proposition 4.1. *Assume that μ is a stationary ergodic elliptic probability measure over cookie environments. If $\mathbb{P}^1(S_+) > 0$, then $\mathbb{P}^1(S_-) = 0$.*

We first prove Theorem 1.2 assuming Proposition 4.1, and then prove Proposition 4.1.

Proof of Theorem 1.2. By Corollary 3.4 we may assume that μ is also non-degenerate. If $\mathbb{P}^1(S_+) > 0$ then by Proposition 4.1 $\mathbb{P}^1(S_-) = 0$ and therefore by Corollary 3.9 $\mathbb{P}_0(X_n \to +\infty) = 1$, if $\mathbb{P}^1(S_-) > 0$ then $\mathbb{P}_0(X_n \to -\infty) = 1$ and otherwise, by Corollary 3.8 $\mathbb{P}_0(|X_n| \to \infty) = 0$ and since μ is non-degenerate, Lemma 3.2 implies $\mathbb{P}_0(X_n = 0 \text{ i.o.}) = 1$. □

Remark 4.2. In some specific cases, e.g. when μ is uniformly elliptic and stationary ergodic or when it is i.i.d. and elliptic, the proof of Proposition 4.1 follows from the symmetry Lemma 2.9 in a much simpler manner.

We now prove Proposition 4.1. We divide the proof of Proposition 4.1 into several steps:

Lemma 4.3. *For every $\epsilon > 0$ there is some y so that $\mathbb{P}^y[S_+] > 1 - \epsilon$.*

Proof. $Z_n^+ > 0$ for all $n \geq 0$ if and only if $T_{-1} = \infty$, and so as in Lemma 3.5, by stationarity and ergodicity of the environment there are a.s. infinitely many optional regeneration positions. In particular there is an a.s. finite (random) first optional regeneration position $\phi > 0$. Consider now the process Z^+ at time ϕ. If Z^+ is positive at time ϕ then it will stay positive forever. Fix $\epsilon > 0$. Let m be a large enough number so that $\mathbb{P}_0^1(\phi > m) < \frac{\epsilon}{2}$. Set $k_m = 1$ and sequentially choose sufficiently large $k_{m-1}, \ldots, k_0 \in \mathbb{N}$ so that $\mathbb{P}(U_{a(j-1,\cdot)}(k_{j-1}) < k_j) < \frac{\epsilon}{2m}$ for all $1 \leq j \leq m$. Setting $y = k_0$, we have

$$\mathbb{P}_0^y[S_+] \geq \mathbb{P}_0^y[Z_\phi^+ \geq 1] \geq \mathbb{P}(U_{a(i-1,\cdot)}(k_{i-1}) \geq k_i, i = 1, ..., m, \phi \leq m) \geq 1 - \epsilon$$

by union bound. □

For a set $B \subset \mathbb{Z}_+$ denote by $\bar{\boldsymbol{\dagger}}(B)$ the upper density of B, that is

$$\bar{\boldsymbol{\dagger}}(B) = \limsup_{n \to \infty} \frac{\#\{j \in B : j < n\}}{n}.$$

Lemma 4.4. $\mathbb{P}^y\left[\bar{\boldsymbol{\dagger}}(\{n : Z_n^+ < k\}) = 0 \mid S_+\right] = 1$ *for every $k \geq 0$.*

Proof. Fix $k > 0$. For $\gamma > 0$ and $x \in \mathbb{Z}$ let $A_{\gamma,x}$ be the event that $\prod_{i=1}^{k}(1-\omega(x,i)) > \gamma$. By stationarity $g(\gamma) := \mu[A_{\gamma,x}]$ is independent of x. By ellipticity, $\lim_{\gamma\to 0} g(\gamma) = 1$. By ergodicity, the set $A_\gamma = \{x : A_{\gamma,x}\}$ has density $g(\gamma)$. Let r be a natural number. Let B_r be the event that $\bar{\mathbb{H}}(\{n : Z_n < k\}) > \frac{1}{r}$ and let γ be small enough so that $g(\gamma) + \frac{1}{r} > 1$. Then, on B_r there are infinitely many n such that both events $Z_n^+ \leq k$ and $A_{\gamma,n+1}$ occur. Let $M_n = \mathbb{P}\big[S_+^c \mid \omega, a(0,\cdot), ..., a(n-1,\cdot)\big]$ where S_+^c is the complement of S_+. Then M_n is a bounded martingale converging $\mathbb{P}$-a.s. to $1_{S_+^c}$. Now since the occurrence of B_r implies that there are infinitely many n for which both events $Z_n^+ \leq k$ and $A_{\gamma,n+1}$ occur, and therefore there are infinitely many n with $M_n \geq \gamma$. In particular, on B_r, $1_{S_+^c} > \gamma$, implying that S_+^c occurs, so $\mathbb{P}\big[B_r \mid S_+\big] = 0$. Since it is true for all $r > 0$, we are done. ☐

Lemma 4.5. *Let $\epsilon > 0$ and let y be from Lemma 4.3. For every $k \geq 0$ there is some l so that $\mathbb{P}^y[Z_l^+ > k] > 1 - 2\epsilon$.*

Proof. Assume not, then there are k and ϵ so that $\mathbb{P}^y\big[Z_l^+ > k \mid S_+\big] < 1 - \epsilon =: \lambda$ for all l. By linearity of expectation, $\mathbb{E}^y\big[\#\{l < n : Z_l^+ > k\} \mid S_+\big] \leq n\lambda$. Fix some $0 < \delta < \lambda$, then by the Markov inequality we have

$$\mathbb{P}^y\Big[\frac{\#\{l < n : Z_l^+ > k\}}{n} > \delta \mid S_+\Big] \leq \frac{\lambda}{\delta}.$$

In other words,

$$\mathbb{P}^y\Big[\frac{\#\{l < n : Z_l^+ \leq k\}}{n} \geq \delta \mid S_+\Big] \geq 1 - \frac{\lambda}{\delta} =: \alpha > 0.$$

But therefore

$$\mathbb{P}^y\Big[\text{ there are infinitely many } n \text{ such that } \frac{\#\{l < n : Z_l^+ \leq k\}}{n} \geq \delta \mid S_+\Big] \geq \alpha,$$

contradicting the fact that conditioned on S_+, $\bar{\mathbb{H}}(\{n : Z_n^+ < k\}) = 0$ $\mathbb{P}^y$-a.s. ☐

By translation invariance of the probability measure μ we get from the symmetry Lemma 2.9 the corollary below. Denote by $\mathbb{P}_r^k[Z_l^- \leq y]$, $r \in \mathbb{Z}$, the probability that on the r-shifted arrow environment $\theta^r a$, the process Z^- with initial value k satisfies $Z_l^- \leq y$.

Corollary 4.6. *For every $k \in \mathbb{N}$, $r_1, r_2 \in \mathbb{Z}$ and $\epsilon > 0$ there is some $l \in \mathbb{N}$ so that $\mathbb{P}_{r_1}^k[Z_l^- \leq y] \geq \mathbb{P}_{r_2}^y[Z_l^+ > k] \geq 1 - 2\epsilon$, where y is as in Lemma 4.3.*

Proof. Fix $k \in \mathbb{N}$ and $\epsilon > 0$ and let l be the one guaranteed in Lemma 4.5. $r_1, r_2 \in \mathbb{Z}$, then the right inequality follows from stationarity of μ, and the left inequality follows from the symmetry Lemma 2.9 and stationarity of μ. ☐

Lemma 4.7. *For every $\epsilon > 0$ there are $n_1 < n_2 < \ldots$ so that $\mathbb{P}^1[Z_{n_i}^- > y] < 3\epsilon$, where again y is as in Lemma 4.3.*

Proof. Fix $m_1 = 0$. There is k_1 so that $\mathbb{P}^1[Z_{m_1}^- > k_1] < \epsilon$. Let l_1 be the l guaranteed by Lemma 4.5 for $k = k_1$. Define $n_1 = m_1 + l_1$, then by Lemma 4.5 $\mathbb{P}^1[Z_{n_1}^- < y] \geq \mathbb{P}^1[Z_{m_1}^- \leq k_1, Z_{n_1}^- < y] \geq 1 - 3\epsilon$. Let $m_2 > n_1$. There is k_2 so that $\mathbb{P}^1[Z_{m_2}^- > k] < \epsilon$ Let l_2 be the l guaranteed by Corollary 4.6 for $k = k_1$, and $t = m_2$. Define $n_2 = m_2 + l_2$, then $\mathbb{P}^1[Z_{n_2}^- < y] \geq \mathbb{P}^1[Z_{m_2}^- \leq k, Z_{n_2}^- < y] \geq 1 - 3\epsilon$. Assume that $n_1 < ... < n_r$ are chosen so that $\mathbb{P}^1[Z_{n_i}^- < y] \geq 1 - 3\epsilon$ for all $1 \leq i \leq r$. At the

$(r+1)$-st step, fix $m_{r+1} > n_r$. There is k_{r+1} so that $\mathbb{P}^1[Z^-_{m_{r+1}} > k] < \epsilon$. Let l_{r+1} be the l guaranteed by Lemma 4.5 for $k = k_{r+1}$ and $t = m_{r+1}$. Define $n_{r+1} = m_{r+1} + l_{r+1}$, then

$$\mathbb{P}^1[Z^-_{n_{r+1}} < y] \geq \mathbb{P}^1[Z^-_{m_{r+1}} \leq k, Z^-_{n_{r+1}} < y] \geq 1 - 3\epsilon.$$

$\square$

Proof of Proposition 4.1. Assume that $\mathbb{P}^1\big[S_+\big] > 0$. Let $\delta > 0$. We will show that $\mathbb{P}^1\big[Z^-_n > 0 \text{ for all } n\big] \leq \delta$. Let $\epsilon = \frac{\delta}{4} > 0$. By Lemma 4.7, there are $n_1 < n_2 < \dots$ so that $\mathbb{P}^1\big[Z^-_{n_i} \leq y\big] \geq 1 - 3\epsilon$. As in the proof of Lemma 4.4, set

$$A_{\gamma,x} = \left\{\prod_{i=1}^{k}(1-\omega(x,i))) > \gamma\right\} \text{ for } \gamma > 0 \text{ and } x \in \mathbb{Z}.$$

By stationarity $g(\gamma) := \mu[A_{\gamma,x}]$ is independent of x. By ellipticity, $\lim_{\gamma\to 0} g(\gamma) = 1$. Let $\gamma > 0$ be small enough so that $g(\gamma) > 1 - \epsilon$. Then $\mathbb{P}^1[Z^-_{n_i} \leq y, A_{\gamma,-n_i}] \geq 1 - 4\epsilon = 1 - \delta$ for all $i \geq 1$. Let D be the event that there are infinitely many i such that $Z^-_{n_i} \leq y$ and $A_{\gamma,-n_i}$. Then $\mathbb{P}^1[D] \geq 1 - \delta$. As before set $M_n = \mathbb{P}[Z^-_l = 0 \text{ for some } l | E, \{(a(0,\cdot), ..., a(i(n-1),\cdot)) : a \in A\}]$, then M_n is a bounded martingale converging to $1_{S^c_-}$, and so if the event D occurs, then also does S^c_-. Therefore

$$\mathbb{P}^1[Z^-_n > 0 \text{ for all } n] \leq 1 - \mathbb{P}^1[D] \leq \delta,$$

and as δ was arbitrary, we are done. $\square$

Acknowledgments

We thank Itai Benjamini, Xiaoqin Guo, Gady Kozma, Igor Shinkar and Ofer Zeitouni for useful discussions. We also thank Jonathon Peterson for reading an earlier version of the manuscript.

References

[1] G. Amir and T. Orenshtein. Excited mob. *In preperation*.

[2] A.L. Basdevant and A. Singh. On the speed of a cookie random walk. *Probability Theory and Related Fields*, 141(3):625–645, 2008.

[3] A.L. Basdevant and A. Singh. Rate of growth of a transient cookie random walk. *Electron. J. Probab*, 13:811–851, 2008.

[4] I. Benjamini and D.B. Wilson. Excited random walk. *Electron. Comm. Probab*, 8(9):86–92, 2003.

[5] Dmitry Dolgopyat. Central limit theorem for excited random walk in the recurrent regime. *ALEA Lat. Am. J. Probab. Math. Stat*, 8:259–268, 2011.

[6] H. Kesten, MV Kozlov, and F. Spitzer. A limit law for random walk in a random environment. *Compositio Math*, 30:145–168, 1975.

[7] E. Kosygina and M.P.W. Zerner. Positively and negatively excited random walks on integers, with branching processes. *Electron. J. Probab*, 13:1952–1979, 2008.

[8] E. Kosygina and M.P.W. Zerner. Excited random walks: results, methods, open problems. *Arxiv preprint arXiv:1204.1895*, 2012.

[9] Elena Kosygina and Thomas Mountford. Limit laws of transient excited random walks on integers. In *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, volume 47, pages 575–600. Institut Henri Poincaré, 2011.

[10] J. Peterson. Strict monotonicity properties in one-dimensional excited random walks. *arXiv preprint arXiv:1210.4518*, 2012.

[11] Jonathon Peterson. Large deviations and slowdown asymptotics for one-dimensional excited random walks. *Electron. J. Probab*, 17(48):1–24, 2012.

[12] Reza Rastegar and Alexander Roitershtein. Maximum occupation time of a transient excited random walk on z. *arXiv preprint arXiv:1111.1254*, 2011.

[13] M.P.W. Zerner. Multi-excited random walks on integers. *Probability Theory and Related Fields*, 133(1):98–122, 2005.